CHALLENGE OF THE WEEK, GRADES 6-7

BLACKSBURG MATH CIRCLE

Make sure that your solution is correct, complete, and clearly written. You should not expect much credit if your proof refers to a false statement, or even if all your statements are true but you forgot to tell us "why?" It is one of the purposes of the Circle to help you improve your "essay-proof" writing style as well as your logical skills.

Please remember that the Challenge is **individual**. Although we strongly encourage cooperation and help among the participants of the Circle, the Weekly Challenge will be one exception to this rule: you may consult your notes, but you may not ask other people to help you.

Problem. Let a, b, c be three integers such that $a+b+c$ is divisible by 30. Prove that $a^5+b^5+c^5$ is divisible by 30.

Solution: In order for $a^5+b^5+c^5$ *to be divisible by* 30 *it is necessary and sufficient that* $a^5+b^5+c^5-(a+b+c)$ *is divisible by* 30*. The latter expression is divisible by* 30 *provided for any integer* n *we have* $30 \mid n^5-n$*. But* $n^5-n=n(n^4-1)=n(n^2-1)(n^2+1)=(n-1)n(n+1)(n^2+1)$*. Since at least one of the numbers* n*,*$n+1$ *is even, this expression is divisible by* 2*. Also, since at least one of the numbers* $n-1$*,* n*, or* $n+1$ *is divisible by* 3 *(why?), we conclude that this expression is also divisible by* 3*. Finally, if neither* $n-1$*,* n*, nor* $n+1$ *are divisible by* 5*, we can deduce that* $n \equiv \pm 2 \pmod 5$ *(why?), and so* $n^2+1 \equiv 4+1 \pmod 5 = 0$*. Therefore,* n^5-n *is divisible by* $2, 3$*, and* 5*, i.e. it is divisible by* 30*.* $\square$

Date: November 5, 2016.